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NONLINEAR MEAN VALUE FORMULAS ON FRACTAL SETS

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Abstract. In this paper we study solutions to nonlinear mean value formulas on fractal sets. We focus on the mean value problem

$$\frac{1}{2}\max_{q\in V_{m,p}}\{f(q)\}+\frac{1}{2}\min_{q\in V_{m,p}}\{f(q)\}-f(p)=0$$

in the Sierpinsky gasket with prescribed values $f(p_1)$, $f(p_2)$ and $f(p_3)$ at the three vertices of the first triangle. For this problem we show existence and uniqueness of a continuous solution and analyze some properties like the validity of a comparison principle, Lipschitz continuity of solutions (regularity) and continuous dependence of the solution with respect to the prescribed values at the three vertices of the first triangle.

1. Introduction.

Our main goal in this paper is to study existence, uniqueness and properties of solutions to mean value formulas when the space ambient is a fractal set.

Mean value formulas appear naturally when one deals with linear PDEs. For example, a well known fact is that u is harmonic in a domain $\Omega\subset\mathbb{R}^N$ (that is u verifies $\Delta u=0$ in Ω) if and only if it verifies the mean value property

$$u(x)=\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)}u(y)\,dy,$$

whenever $B_\varepsilon(x)\subset\Omega$. One surprising fact is that mean value properties also appear in connection with nonlinear problems. In fact, in [1] it is proved that a function u is $\infty-$harmonic (that is u verifies $\Delta_\infty u=\langle D^2u\nabla u;\nabla u\rangle=0$) if and only if the following asymptotic mean value formula holds (in the viscosity sense),

$$u(x)=\frac{1}{2}\left\{\max_{\overline{B_\varepsilon(x)}}u+\min_{\overline{B_\varepsilon(x)}}u\right\}+o(\varepsilon^2),\quad \text{as } \varepsilon\to 0.$$

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See also [1] for a mean value characterization of $p-$harmonic functions, that is, solutions to $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$. For more references on this subject we quote [2, 3, 4, 5, 6]. Some of these mean value formulas are related with numerical methods, [7, 8].

These mean value formulas are closely related to game theory. In particular, in [9] a game called *Tug-of-War* is studied in connection with the $\infty-$Laplacian, $\Delta_\infty u = \langle D^2 u \nabla u; \nabla u\rangle$. This is a two-player, zero-sum game in which a fair coin is tossed (with probabilities 1/2 - 1/2) and the winer of the coin toss chooses the new position of the game in the ball of radius ε centered at the previous position. Then, the value function of the game u^ε (the expected payoff for both players) verifies the mean value formula

$$u^\varepsilon(x) = \frac{1}{2}\sup_{B_\varepsilon(x)} u^\varepsilon + \frac{1}{2}\inf_{B_\varepsilon(x)} u^\varepsilon. \tag{1.1}$$

We refer to [9] and [10] for a detailed study of this game. We remark that taking the limit as $\varepsilon \to 0$ the u^ε converge to the solution to $\Delta_\infty u = \langle D^2 u \nabla u; \nabla u\rangle = 0$ (notice the connection with the mean value characterization).

On the other hand, mean value formulas in discrete settings were also studied. Here we refer to [11, 12, 13, 14, 15, 16, 17, 18, 19] and references therein, for mean value formulas on trees. Trees are prototypes of simple ambient spaces where one can investigate mean value formulas that are similar to the ones studied here.

In addition, for fractal sets there is a well known Laplacian (linear) operator, see the book [20] and [21, 22]. The Laplacian in this context appears from the work of several probabilists who constructed stochastic processes analogous to Brownian motion, thus obtaining a Laplacian indirectly as the generator of the process. See the book [23]. Recently, linear mean value properties for harmonic function were investigated in [24]. Concerning the infinity laplacian on fractal sets we quote the recent paper [25] where an infinity laplacian is constructed on the Sierpinski gasket in connection with absolutely minimizing Lipschitz extensions of a given datum. See also [26] for more references on equations on SG.

As we have mentioned, the purpose of this paper is to initiate the study of solutions to nonlinear mean value formulas on fractals. We restrict our attention to the prototype of all fractals, the Sierpinski gasket (called SG along this paper). Some of our results are generic and extend easily to Kigami's class of post-critically finite (PCF) fractals [27, 20]. But we confine ourselves to SG since some of our results are based specifically in the geometry of SG.

Let us introduce SG (for a more detailed definition, see Section 2). Let $\{p_1, p_2, p_3\}$ be the vertices of an equilateral triangle of unit length in the plane. For $i = 1, 2, 3$, define a contraction F_i of the plane by

$$F_i(x) = \frac{1}{2}(x - p_i) + p_i,$$

Now, let $V_0 = \{p_1, p_2, p_3\}$, and, for each integer $m \geq 1$,

$$V_m = \bigcup_{w \in \{1,2,3\}^m} F_w(V_0),$$

where $F_w(V_0) = F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_m}(V_0)$ if $w = i_1 i_2 \ldots i_m \in \{1, 2, 3\}^m$. Furthermore, we write $F_w(V_0)$ as $\{p_1(w), p_2(w), p_3(w)\}$. For $p \in V_m$ we define $V_{m,p}$ as the set

$$V_{m,p} = \left\{ \begin{array}{c} q \in V_m : q \text{ is connected to } p \text{ by a side of a triangle} \\ \text{with vertices in } F_w(V_0) \end{array} \right\}.$$

Set

$$V_* = \bigcup_{m \geq 0} V_m.$$

Then the closure $\overline{V}_*$ is the Sierpinski gasket, SG. See Figure 1.

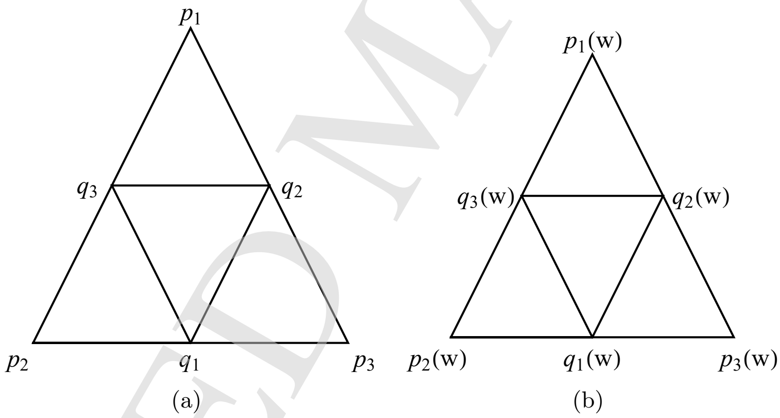

FIGURE 1. The first step in the construction of SG (a) and a zoom of the n–th iteration (b).

Now we introduce the discrete mean value formula that we want to study on the Sierpinsky gasket. The definition that we give below corresponds to a max − min mean value formula in the spirit of (1.1), see [9].

Problem. We prescribe $f(p_1)$, $f(p_2)$ and $f(p_3)$, three values for the vertices p_1, p_2 and p_3 (without loss of generality, we will assume that $f(p_1) \leq f(p_2) \leq f(p_3)$). Our aim is to look for continuous functions $f \in C(SG)$, that verify the mean value property

$$(L_m^\infty f)(p) := \frac{1}{2} \max_{q \in V_{m,p}} \{f(q)\} + \frac{1}{2} \min_{q \in V_{m,p}} \{f(q)\} - f(p) = 0$$

for every $m \geq 1$ and $p \in V_m \setminus V_{m-1}$ and they take the prescribed values $f(p_1)$, $f(p_2)$ and $f(p_3)$ at the three vertices p_1, p_2 and p_3. We will refer to this problem as a Dirichlet problem with boundary conditions $f(p_1)$, $f(p_2)$, $f(p_3)$.

For this problem we have a game theoretic interpretation in the same spirit of [9] and [10]: here, the *Tug-of-War* game described before is played (a two-player, zero-sum game in which a fair coin is tossed (with probabilities 1/2 - 1/2) and the winer of the coin toss chooses the new position of the game) with the restriction that at a vertex p of a triangle of the SG the next position of the game can be chosen only among the vertices of triangles of SG that are connected to p, that is, $q \in V_{m,p}$. Then, the Dynamic Programming Principle for this game says that the value function verifies

$$\frac{1}{2} \max_{q \in V_{m,p}} \{f(q)\} + \frac{1}{2} \min_{q \in V_{m,p}} \{f(q)\} - f(p) = 0$$

for every $m \geq 1$ and $p \in V_m \setminus V_{m-1}$. That is, the value of this game is a solution to our equation on the vertices of the triangles of the SG. For a proof of this fact we refer to [28].

We have existence, uniqueness and the validity of a comparison principle for our problem.

Theorem A. *Given three numbers $\alpha \leq \beta \leq \gamma$, there exists a unique continuous function $f : SG \mapsto \mathbb{R}$ such that*

$$(L^\infty_m f)(p) = \frac{1}{2} \max_{q \in V_{m,p}} \{f(q)\} + \frac{1}{2} \min_{q \in V_{m,p}} \{f(q)\} - f(p) = 0$$

for every $m \geq 1$ and $p \in V_m \setminus V_{m-1}$, and satisfies $f(p_1) = \alpha$, $f(p_2) = \beta$, and $f(p_3) = \gamma$.

Moreover, the following properties hold:

- *The solution f is Lipschitz with Lipschitz constant less or equal than $2\max_{1 \leq i < j \leq 3}\{|f(p_i) - f(p_j)|\}$.*

- *There is a strong maximum principle; if a solution f attains its maximum value in the interior of SG, that is, in $SG \setminus V_0$, then f is constant in SG.*

- *There is a comparison principle; if $f(p_1) \leq g(p_1)$, $f(p_2) \leq g(p_2)$ and $f(p_3) \leq g(p_3)$ and f and g denote the corresponding solutions, then we have that*

$$f(p) \leq g(p), \qquad \textit{for every } p \in SG.$$

Now, we state a curious geometric property of the solution. We have that solutions are linear on a large collection of sides of the triangles that compose SG. In every triangle there is a side on which the solution is linear. However, it is not true that the solution is linear function on the whole SG (notice that the mean value formula under consideration is highly nonlinear).

Theorem B. *Let f be the solution in SG satisfying the boundary conditions $f(p_1) \le f(p_2) \le f(p_3)$, then f is lineal on the segment with endpoints p_1 and p_3, $L_{p_1p_3}$. This linearity also holds on every side of a triangle in SG that joins the maximum and the minimum of f in this triangle.*

Finally, we have continuous dependence of the solutions with respect to the boundary conditions $f(p_1)$, $f(p_2)$ and $f(p_3)$. Therefore, our problem is well posed (we have existence, uniqueness and continuous dependence with respect to the datum).

Theorem C. *Let $(g_n(p_1))_n$, $(g_n(p_2))_n$ and $(g_n(p_3))_n$ three sequences converging to $f(p_1)$, $f(p_2)$ and $f(p_3)$ respectively. For each $n \in \mathbb{N}$ let g_n be the solution in SG with boundary conditions $g_n(p_1)$, $g_n(p_2)$ and $g_n(p_3)$ and let f be the solution in SG with boundary conditions $f(p_1)$, $f(p_2)$ and $f(p_3)$. Then, we have that*

$$\lim_{n\to\infty} g_n = f$$

uniformly in SG.

To end this introduction let us make a brief comment comparing our results with the ones that hold for the standard mean value

$$(L_m^2 f)(p) = \frac{\sum_{q\in V_m(p)} f(q)}{\sharp V_m(p)} - f(p) = 0.$$

Notice that this operator L_m^2 is linear. For this problem it is well known, see the book [20] and [21, 22, 24], that there is existence and uniqueness of solutions (in the same spirit as in Theorem A) and continuous dependence with respect to the data (like in Theorem C). However, surprisingly, the linear structure that is behind the nonlinear mean value formula studied here (Theorem B) does not appear for L_m^2.

The paper is organized as follows: In Section 2 we collect two equivalent definitions of SG; in Section 3 we prove Theorem A; in Section 4 we analyze the finner structure of the solutions and prove Theorem B; finally, Section 5 is devoted to the proof of continuous dependence of the solutions with respect to the data, Theorem C.

2. Definitions.

We begin defining the Sierpinski gasket of two different ways. First we will use the theory of Iterated System Functions (IFS), while the second will be the one that helped us to define our problem in the introduction.

Definition 2.1. Let $\{p_1, p_2, p_3\}$ be the vertices of an equilateral triangle of unit length in the plane. For $i = 1, 2, 3$, define a contraction F_i of the plane by

$$F_i(x) = \frac{1}{2}(x - p_i) + p_i,$$

which has the unique fixed point p_i. Then the contractions $\{F_1, F_2, F_3\}$ defines a self-similar set SG (the Sierpinski gasket) as the unique compact satisfying

$$SG = F_1(SG) \cup F_2(SG) \cup F_3(SG).$$

The second definition is just the one that we used in the introduction.

Definition 2.2. With the same notations as in the previous definition, let $V_0 = \{p_1, p_2, p_3\}$, and, for each integer $m \geq 1$,

$$V_m = \bigcup_{w \in \{1,2,3\}^m} F_w(V_0),$$

where $F_w(V_0) = F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_m}(V_0)$ if $w = i_1 i_2 \ldots i_m \in \{1,2,3\}^m$. Furthermore, we write $F_w(V_0)$ as $\{p_1(w), p_2(w), p_3(w)\}$. Set

$$V_* = \bigcup_{m \geq 0} V_m.$$

Then, the closure $\overline{V}_*$ is the Sierpinski gasket SG.

3. Existence and uniqueness of solutions and some properties.

First, we show that, for each boundary condition, there is only one continuous solution to our problem.

Proof of existence and uniqueness of solutions. Our aim is to show that, given three numbers $\alpha \leq \beta \leq \gamma$, there exists a unique continuous solution f to

$$(L_m^\infty f)(p) = \frac{1}{2} \max_{q \in V_{m,p}} \{f(q)\} + \frac{1}{2} \min_{q \in V_{m,p}} \{f(q)\} - f(p) = 0$$

with $f(p_1) = \alpha$, $f(p_2) = \beta$, and $f(p_3) = \gamma$.

Let us start with the first step, that is, we look for the solution in $V_1 \setminus V_0$. To this end, we write $(L_1^\infty f)|_{V_1 \setminus V_0}$ in the matrix form. We have to distinguish three different cases. Let

(3.1)
$$A_1 = \frac{1}{4} \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}, A_2 = \frac{1}{6} \begin{pmatrix} 2 & 0 & 4 \\ 3 & 0 & 3 \\ 4 & 0 & 2 \end{pmatrix}, A_3 = \frac{1}{4} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

Now the solution to our problem in $V_1 \setminus V_0$ is given by

$$\begin{pmatrix} f(q_1) \\ f(q_2) \\ f(q_3) \end{pmatrix} = A \begin{pmatrix} f(p_1) \\ f(p_2) \\ f(p_3) \end{pmatrix},$$

where

$$A = \begin{cases} A_1 & \text{if } 3f(p_2) < 2f(p_1) + f(p_3), \\ A_2 & \text{if } 2f(p_1) + f(p_3) \le 3f(p_2) \le f(p_1) + 2f(p_3), \\ A_3 & \text{if } f(p_1) + 2f(p_3) < 3f(p_2). \end{cases}$$

We have thus determined the values of f in V_1. Now, proceeding by induction, we assume that we have the values of f on V_m and look for the values in V_{m+1}.

If we look at $F_w(V_0) = \{p_1(w), p_2(w), p_3(w)\}$ then by the same arguments we obtain the values in V_{m+1}. In fact, if we want $(L^\infty_{m+1})(q_i(w)) = 0$, we can conclude solving exactly the same system as the one that we solved to obtain the values of f on V_1 from its values on V_0. We obtain

$$\begin{pmatrix} f(q_1(w)) \\ f(q_2(w)) \\ f(q_3(w)) \end{pmatrix} = A \begin{pmatrix} f(p_1(w)) \\ f(p_2(w)) \\ f(p_3(w)) \end{pmatrix}.$$

In this way we can determine inductively, starting with the values of f in V_0, the values on V_m for all $m \in \mathbb{N}$. If we define $V_* = \cup_{m\ge1} V_m$, we can find $f : V_* \to \mathbb{R}$. We complete the proof by showing the following:

(1) f satisfies $(L^\infty_m f)(p) = 0$ for every $m \ge 1$ and every $p \in V_m \setminus V_{m-1}$.
(2) We can extend f to a continuous function on SG.

Proof of (1). It follows from the previous construction.
Proof of (2). We observe the following facts: For q_j, $j = 1, 2, 3$,

$$\min\{f(p_i); i = 1, 2, 3\} \le f(q_j) \le \max\{f(p_i); i = 1, 2, 3\}.$$

This means that each new value is between the maximum and minimum of the above gasket.

For $k = 1, 2, 3$,

$$\max_{1\le i<j\le3} \{|f(p_i(k)) - f(p_j(k))|\} \le \frac{1}{2} \max_{1\le i<j\le3} \{|f(p_i) - f(p_j)|\}.$$

We use these estimates to show (by induction) that for any $w \in \{1, 2, 3\}^m$ it holds that

(1) For $p \in V_* \cap F_w(SG) = V_* \cap SG_w$,

$$\min\{f(p_i); i = 1, 2, 3\} \le f(p) \le \max\{f(p_i); i = 1, 2, 3\}.$$

(2) If we denote

$$v_w(f) = \max_{1\le i<j\le3} \{|f(p_i(w)) - f(p_j(w))|\},$$

we have

$$v_w(f) \le \max_{1\le i<j\le3} \{|f(p_i) - f(p_j)|\} \frac{1}{2^m} = C \frac{1}{2^m}.$$

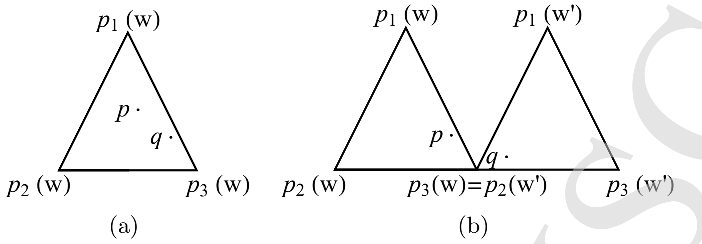

FIGURE 2

Now, select two points $p, q \in V_*$ such that $|p-q| \leq \frac{1}{2^m}$. Recall that we started the construction of the Sierpinski gasket with the equilateral triangle of unit length whose vertices were $\{p_1, p_2, p_3\}$, so that for any $w \in \{1,2,3\}^m$ the diameter of SG_w is $(\frac{1}{2})^m$. Thus when we have $|p-q| \leq \frac{1}{2^m}$, the following two possibilities arise (see Figure 2).

a) For some $w \in \{1,2,3\}^m$ we have $p, q \in SG_w$.
b) For some $w, w' \in \{1,2,3\}^m$ such that $SG_w \cap SG_{w'} \neq \emptyset$ we have, $p \in SG_w$, $q \in SG_{w'}$.

If a) occurs, then our previous estimates imply that $|f(p) - f(q)| \leq C\frac{1}{2^k}$. Similarly, if we are in case b) we can show that $|f(p) - f(q)| \leq 2C\frac{1}{2^k}$. Hence we have proved that for any two points $p, q \in V_*$ satisfying $|p - q| \leq \frac{1}{2^k}$, it holds that

$$|f(p) - f(q)| \leq 2C\frac{1}{2^k}.$$

Now given $x \in SG$, we choose a sequence $(x_n)_{n\geq 1}$ in V_* converging to x as $n \to +\infty$. Then, for any k we can find large enough m and n such that

$$|x_n - x_m| \leq \frac{1}{2^k}.$$

So we have that

$$|f(x_n) - f(x_m)| \leq 2C\frac{1}{2^k}.$$

Hence $(f(x_n))_{n\geq 1}$ is a Cauchy sequence, and then it converges as $n \to +\infty$. If x does not belong to V_* we define the value of f at x by

$$f(x) = \lim_{n\to+\infty} f(x_n).$$

In this way f is extended to a function on SG. The continuity of f follows from this construction. $\square$

Definition 3.2. Under the previous conditions we say that the function f has boundary datum of type I if $3f(p_2) < 2f(p_1) + f(p_3)$, type II if $2f(p_1)+f(p_3) \leq 3f(p_2) \leq f(p_1)+2f(p_3))$, and of type III if $f(p_1)+2f(p_3) < 3f(p_2)$.

Remark 3.3. The boundary conditions tell us that we will have three different kind of solutions depending on the fact that the value $f(p_2)$ stays in the first, in the second or in the third third of the interval $[f(p_1), f(p_3)]$.

Corollary 3.4. *If f is a solution in SG, then f is Lipschitz with Lipschitz constant less or equal than $2C = 2\max_{1\le i<j\le 3}\{|f(p_i) - f(p_j)|\}$.*

Proof. It follows from our construction. □

Our next result is the validity of a strong maximum principle for our problem. To proof this principle we will use next lemma.

Lemma 3.5. *Let f be a solution in K satisfying $f(p_1) \le f(p_2) \le f(p_3)$. We have*

$$\frac{2f(p_1)+f(p_2)+f(p_3)}{4} \le f(q_i) \le \frac{f(p_1)+f(p_2)+2f(p_3)}{4}, i = 1,2,3. \tag{3.6}$$

Proof. We know that $f(q_3) \le f(q_2) \le f(q_1)$ and we have to show that

$$\frac{2f(p_1)+f(p_2)+f(p_3)}{4} \le f(q_3), \text{ and } f(q_1) \le \frac{f(p_1)+f(p_2)+2f(p_3)}{4}.$$

Thanks to (3.1) we obtain

$$f(q_3) = \begin{cases} \frac{1}{4}(2f(p_1)+f(p_2)+f(p_3)) \text{ if } 3f(p_2) < 2f(p_1)+f(p_3), \\ \frac{1}{3}(2f(p_1)+f(p_3)) \text{ if } 3f(p_2) \in [2f(p_1)+f(p_3), f(p_1)+2f(p_3)], \\ \frac{1}{2}(f(p_1)+f(p_2)) \text{ if } f(p_1)+2f(p_3) < 3f(p_2). \end{cases}$$

and

$$f(q_1) = \begin{cases} \frac{1}{2}(f(p_2)+f(p_3)) \text{ if } 3f(p_2) < 2f(p_1)+f(p_3), \\ \frac{1}{3}(f(p_1)+2f(p_3)) \text{ if } 2f(p_1)+f(p_3) \le 3f(p_2) \le f(p_1)+2f(p_3), \\ \frac{1}{4}(f(p_1)+f(p_2)+2f(p_3)) \text{ if } f(p_1)+2f(p_3) < 3f(p_2). \end{cases}$$

Now

$$\frac{1}{4}(2f(p_1)+f(p_2)+f(p_3)) \le \frac{1}{3}(2f(p_1)+f(p_3))$$

if and only if

$$3f(p_2) < 2f(p_1)+f(p_3),$$

and

$$\frac{1}{3}(2f(p_1)+f(p_3)) \le \frac{1}{2}(f(p_1)+f(p_2))$$

if and only if

$$f(p_1)+2f(p_3) < 3f(p_2),$$

and thus we conclude

$$\frac{1}{4}(2f(p_1)+f(p_2)+f(p_3)) \le f(q_3) \le \frac{1}{2}(f(p_1)+f(p_2)). \tag{3.7}$$

In a similar way we have

$$\frac{1}{2}(f(p_2)+f(p_3)) \le \frac{1}{3}(f(p_1)+2f(p_3))$$

if and only if

$$3f(p_2) < 2f(p_1)+f(p_3),$$

and

$$\frac{1}{3}(f(p_1)+2f(p_3)) \le \frac{1}{4}(f(p_1)+f(p_2)+2f(p_3))$$

if and only if

$$f(p_1)+2f(p_3) < 3f(p_2).$$

Therefore

$$\frac{1}{2}(f(p_2)+f(p_3)) \le f(q_1) \le \frac{1}{4}(f(p_1)+f(p_2)+2f(p_3)). \tag{3.8}$$

Joining (3.7) and (3.8) we finish the proof since we get

$$\begin{aligned}\frac{1}{4}(2f(p_1)+f(p_2)+f(p_3)) &\le f(q_3) \le f(q_2) \\ &\le f(q_1) \le \frac{1}{4}(f(p_1)+f(p_2)+2f(p_3)),\end{aligned}$$

as we wanted to show. □

Proposition 3.9. *If a solution f, defined on SG, attains its maximum value in the interior of SG, that is, in $SG\setminus V_0$, then f is constant in SG.*

Proof. Assume that f attains its maximum at some interior point x^*, that is, $f(x^*) = \max\{f(x) : x \in SG\setminus V_0\} = \max_{SG} f$. W If we have that $f(p_1) \le f(p_2) \le f(p_3)$ (without loss of generality we can assume this), we know that $f(x^*) = f(p_3)$ and then we have three possibilities

- $x^* \in F_1(SG)$. Thus from Lemma 3.5 we get
$$f(x^*) = f(p_3) \le \frac{1}{4}(f(p_1)+f(q_3)+2f(q_2)) \to f(p_1) = f(p_3).$$
- $x^* \in F_2(SG)$. In this case thanks again to Lemma 3.5 we have
$$f(x^*) = f(p_3) \le \begin{cases} \frac{1}{4}(f(q_3)+f(q_1)+2f(p_2)), \\ \text{or} \\ \frac{1}{4}(f(q_3)+f(p_2)+2f(q_2)). \end{cases}$$
so, $f(q_3) = f(p_3) = \frac{1}{2}(f(p_1)+f(p_3))$ and thus $f(p_1) = f(p_3)$.

- $x^* \in F_3(SG)$. In this case we obtain
$$f(x^*) = f(p_3) \leq \frac{1}{4}(f(q_2) + f(q_1) + 2f(p_3))$$
thus $f(p_2) = f(p_3) = \frac{1}{2}(f(p_1) + f(p_3))$ and so $f(p_1) = f(p_3)$.

In any case we obtain
$$f(p_1) = f(p_2) = f(p_3),$$
and then we conclude that f is constant in SG. □

Corollary 3.10. *If f is a solution on SG, with nonnegative boundary conditions $0 \leq f(p_1) \leq f(p_2) \leq f(p_3)$ and exists $x \in SG$ such that $f(x) = 0$ then f is identically zero on SG.*

Proof. The same argument used to prove the Proposition 3.9 works here. □

Next, we observe that we have a comparison principle for solutions.

Theorem D. *If f and g are solutions on SG with*
$$f(p_1) \leq g(p_1), f(p_2) \leq g(p_2) \text{ and } f(p_3) \leq g(p_3),$$
then
$$f(x) \leq g(x)$$
for every $x \in SG$.

The proof of this result is obtained of the following three lemmas. The case when f and g have the same boundary conditions is trivial, hence we can assume that we have a strict inequality. Arguing by induction we concentrate only in the first step and show that $f(q_1) \leq g(q_1)$, $f(q_2) \leq g(q_2)$ and $f(q_3) \leq g(q_3)$. Notice that, under our conditions, we have that $f(q_2) \leq g(q_2)$ (this follows from our construction of the solution), so in the next lemmas we will get the other two inequalities.

Lemma 3.11. *Under the hypotheses of Theorem D, if f has boundary conditions of type I then $f(q_1) \leq g(q_1)$ and $f(q_3) \leq g(q_3)$.*

Proof. We know that $f(p_2) < 2f(p_1) + f(p_3)$. We will start by q_1,
$$f(q_1) = \frac{1}{2}(f(p_2) + f(p_3)) = \frac{1}{12}(6f(p_2) + 6f(p_3)) \leq \frac{1}{12}(4f(p_1) + 8f(p_3)).$$
Now,
$$f(q_1) \leq \frac{1}{3}(f(p_1) + 2f(p_3)) \leq g(q_1),$$
if g has initial conditions of type II and
$$f(q_1) \leq \frac{1}{12}(3f(p_1) + f(p_1) + 2f(p_3) + 6f(p_3))$$
$$\leq \frac{1}{4}(g(p_1) + g(p_2) + 2g(p_3)) = g(q_1),$$

if g has initial conditions of type III.

The inequality at q_3 is similar. We have

$$f(q_3) = \frac{1}{4}(2f(p_1) + f(p_2) + f(p_3)) = \frac{1}{12}(8f(p_1) + 4f(p_3)).$$

So,

$$f(q_3) \le \frac{1}{3}(2f(p_1) + f(p_3)) \le g(q_3),$$

if g has initial conditions of type II, and

$$f(q_3) \le \frac{1}{2}(6f(p_1) + 6f(p_3)) \le g(q_3),$$

if g has initial conditions of type III. □

The next lemma tackles the case in which f has boundary conditions of type II.

Lemma 3.12. *Under the hypotheses of Theorem D, if f has boundary conditions of type II then $f(q_1) \le g(q_1)$ and $f(q_3) \le g(q_3)$.*

Proof. We know that $2f(p_1) + f(p_3) \le 3f(p_2) \le f(p_1) + 2f(p_3)$. If we take q_1 we obtain,

$$f(q_1) = \frac{1}{3}(f(p_1) + 2f(p_3)) = \frac{1}{12}(4f(p_2) + 8f(p_3)).$$

Now, if g has initial conditions of type I we have

$$f(q_1) \le \frac{1}{12}(2(2f(p_1) + f(p_3)) + 6f(p_3)) \le \frac{1}{12}(6f(p_2) + 6f(p_3)) \le g(q_1),$$

and if g has initial conditions of type III we get

$$\begin{aligned} f(q_1) &\le \frac{1}{12}(3g(p_1) + f(p_1) + 2f(p_3) + 6g(p_3)) \\ &\le \frac{1}{12}(3g(p_1) + 3g(p_2) + 6g(p_3)) \le g(q_1). \end{aligned}$$

For q_3 we have,

$$f(q_3) = \frac{1}{3}(2f(p_1) + f(p_3)) = \frac{1}{12}(8f(p_1) + 4f(p_3)).$$

So,

$$f(q_3) \le \frac{1}{12}(6f(p_1) + 3f(p_2) + 3f(p_3)) \le g(q_3),$$

if g has initial conditions of type I, and

$$f(q_3) \le \frac{1}{12}(6f(p_1) + 2(f(p_1) + 2f(p_3))) \le \frac{1}{12}(6g(p_1) + 6g(p_2)) \le g(q_3),$$

if g has initial conditions of type III. □

Below we include the last lemma in order to conclude the proof of Theorem D.

Lemma 3.13. *Under the hypotheses of Theorem D, if f has boundary conditions of type III then $f(q_1) \le g(q_1)$ and $f(q_3) \le g(q_3)$.*

Proof. We know that $f(p_1) + 2f(p_3) < 3f(p_2)$. For q_1 we obtain,

$$f(q_1) = \frac{1}{4}(f(p_1) + f(p_2) + 2f(p_3)).$$

Now, if g has initial conditions of type I we have

$$f(q_1) \le \frac{1}{2}(f(p_2) + f(p_3)) \le g(q_1),$$

and if g has initial conditions of type II we get

$$f(q_1) \le \frac{1}{12}(3g(p_1) + 3g(p_2) + 6g(p_3)) \le \frac{1}{12}(4g(p_1) + 8g(p_3)) \le g(q_1).$$

Now if we take q_3 we have,

$$f(q_3) = \frac{1}{2}(f(p_1) + f(p_2)).$$

So,

$$f(q_3) \le \frac{1}{4}(2f(p_1) + 2f(p_2) + f(p_3)) \le g(q_3),$$

if g has initial conditions of type I, and

$$f(q_3) \le \frac{1}{6}(3g(p_1) + 3g(p_2)) \le \frac{1}{3}(2g(p_1) + g(p_3)) = g(q_3),$$

if g has initial conditions of type II. □

Example 3.14. Let f and g be two solutions in SG satisfying the boundary conditions

$$f(p_1) = 0 = g(p_1) \le f(p_2) = 1 = g(p_2) \le f(p_3) = \gamma_1 < g(p_3) = \gamma_2 < \frac{3}{2}.$$

It is easy to see that

$$f(q_3) = \frac{1}{2} = g(q_3).$$

With this simple example we show that the strong comparison principle does not hold. Two ordered solutions can be equal in an interior point but differ on the boundary.

The lack of strong comparison principle can be even more delicate as the following example shows.

Example 3.15. Let f and g be two solutions in SG satisfying the boundary conditions

$$f(p_1) = 0 = g(p_1) \le f(p_2) = 4 < 5 = g(p_2) < f(p_3) = 9 = g(p_3).$$

In this case we have

$$f(p_2(1)) = 3 = g(p_2(1)),$$
$$f(p_3(1)) = \frac{9}{2} = g(p_3(1))$$
and
$$f(p_2(3)) = 6 = g(p_2(3)),$$

so f take the same values on all points in SG such that $x \in SG_1 \cup SG_3$.

4. Structure of the solutions

If we denote by T the triangle of vertices $\{p_1, p_2, p_3\} = V_0$, we observe that after the first stage we built our solution in the vertices of the three new (smaller) triangles $T_i = F_i(T)$. The vertices are given by $F_i(V_0)$, for $i = 1, 2, 3$. In this way when we obtained the solution f on SG we first compute the values in $V_1 \setminus V_0$, after this we repeat this process and compute in $V_2 \setminus V_1$, etc. To obtain f in $V_1 \setminus V_0$ we needed to know what type of initial conditions we start. If we fix the boundary conditions, we observe that to know the values of f in $V_2 \setminus V_1$ we have to solve three numerical systems. The problem is that, a priori, we do not know of what type they are. In this section, we will study how the first boundary conditions determine the type of the following triangles.

Definition 4.1. We say that a triangle T of vertices $\{p_1, p_2, p_3\}$ is type I if f has boundary conditions of type I. That is if $3f(p_2) < 2f(p_1) + f(p_3)$. The other types are similarly defined.

To simplify the notation, if we exclude the trivial case $f(p_1) = f(p_2) = f(p_3)$, we can write

$$f(p_1) = x, \quad f(p_2) = x + my, \quad y > 0, m \in [0, 1] \text{ and } f(p_3) = x + y.$$

Furthermore, we can rewrite the above definition by saying that $T = \{p_1, p_2, p_3\}$ is type I, II, or III according to the value of $m \in [0, 1]$, and therefor we will say that $T = \{p_1, p_2, p_3\}$is Type I, II or III with order $m \in [0, 1]$.

Lemma 4.2. *Let $T = \{p_1, p_2, p_3\}$ be a triangle.*

(1) T is type I if and only if $m \in [0, \frac{1}{3})$.
(2) T is type II if and only if $m \in [\frac{1}{3}, \frac{2}{3}]$.
(3) T is type III if and only if $m \in (\frac{2}{3}, 1]$.

Proof. *(1)* T is type I if and only if $3f(p_2) < 2f(p_1) + f(p_3)$ or equivalently with the new notation

$$3(x + my) < 2x + x + y \iff m \in [0, \frac{1}{3}).$$

(2) In this case we have that $2f(p_1) + f(p_3) \le 3f(p_2) \le f(p_1) + 2f(p_3)$. So,

$$2x + x + y \le 3x + 3my \le x + 2x + 2y \iff m \in [\frac{1}{3}, \frac{2}{3}].$$

(3) T is type III then we know that $f(p_1) + 2f(p_3) < 3f(p_2)$, therefore

$$x + 2x + 2y < 3x + 3my \iff m \in (\frac{3}{2}, 1].$$

This ends the proof. $\square$

Next, we will study the behavior of each type of boundary conditions when we iterate our construction.

4.1. **Boundary conditions of type** I. We will work with a solution f that verifies $f(p_2) < 2f(p_1) + f(p_3)$ or equivalently with $m < 1/3$ thanks to Lemma 4.2. We will show how the type of T affects the type of $T_i = F_i(T)$, $i = 1, 2, 3$.

Lemma 4.3. *If $T = \{p_1, p_2, p_3\}$ is type I (that is, $m \in [0, \frac{1}{3})$) then*

(1) $F_1(T) = T_1 = \{F_1(p_1), F_1(p_2), F_1(p_3)\} = \{p_1, q_3, q_2\}$ *is type II with*
$$m_1 = \frac{m+1}{2} \in [\frac{1}{2}, \frac{2}{3}).$$
(2) $F_2(T) = T_2 = \{F_2(p_2), F_2(p_1), F_2(p_3)\} = \{p_2, q_3, q_1\}$ *has*
$$m_2 = \frac{1}{2}(\frac{1-3m}{1-m}) \in (0, \frac{1}{2}].$$
Therefore, T_2 is type II if $0 \le m \le \frac{1}{7}$ and it is type I if $\frac{1}{7} < m < \frac{1}{3}$.

(3) $F_3(T) = T_3 = \{F_3(p_1), F_3(p_2), F_3(p_3)\} = \{q_2, q_1, p_3\}$ *is type I with*
$$m_3 = m.$$

Proof. We know that

$$f(q_1) = x + \frac{m+1}{2}y, \quad f(q_2) = x + \frac{1}{2}y, \text{ and } f(q_3) = x + \frac{m+1}{4}y,$$

so

$$x \le x + (\frac{m+1}{2})\frac{y}{2} \le x + \frac{y}{2},$$

and then T_1 is of order $\frac{m+1}{2} \in [\frac{1}{2}, \frac{2}{3})$, thus T_1 is type II.

Now, we deduce that the order of T_2 is

$$x + my \le x + my + \frac{1}{2}(\frac{1-3m}{1-m})\frac{1-m}{2}y \le x + my + \frac{1-m}{2}y,$$

so T_2 is type $m_2 = \frac{1}{2}(\frac{1-3m}{1-m})$ and we conclude that if $m \in [0, \frac{1}{7}]$ then $m_2 \in [\frac{1}{3}, \frac{1}{2}]$ and thus T_2 is type II, on the other hand if $m \in (\frac{1}{7}, \frac{1}{3})$ then T_2 is type I since $m_2 \in (0, \frac{1}{3})$.

Finally, for $T_3 = \{q_2, q_1, p_3\}$ we have

$$x + \frac{y}{2} \leq x + \frac{y}{2} + m\frac{y}{2} \leq x + \frac{y}{2} + \frac{y}{2},$$

then T_3 is type I with m fixed. □

4.2. Boundary conditions of type II.

Lemma 4.4. *If $T = \{p_1, p_2, p_3\}$ is type II of order $m \in [\frac{1}{3}, \frac{2}{3}]$ then*

(1) $T_1 = \{p_1, q_3, q_2\}$ is type II of order

$$m_1 = \frac{2}{3}.$$

(2) $T_2 = \{q_3, p_2, q_1\}$ is of order

$$m_2 = 3m - 1 \in [0, 1].$$

Therefore
(a) If $m \in [\frac{1}{3}, \frac{4}{9})$ then T_2 is type I.
(b) If $m \in [\frac{4}{9}, \frac{5}{9}]$ then T_2 is type II.
(c) If $m \in (\frac{5}{9}, \frac{2}{3}]$ then T_2 is type III.

(3) $T_3 = \{q_2, q_1, p_3\}$ is type II of order

$$m_3 = 1/3.$$

Proof. We have

$$f(q_1) = x + \frac{2}{3}y, \quad f(q_2) = x + \frac{1}{2}y, \text{ and } f(q_3) = x + \frac{1}{3}y.$$

For T_1 we observe that

$$x < x + \frac{2}{3}\frac{y}{2} < x + \frac{y}{2},$$

therefore T_1 is type II with order $m_1 = \frac{2}{3}$.

Now, T_2 verifies

$$x + \frac{y}{3} < x + \frac{y}{3} + (3m - 1)\frac{y}{3} < x + \frac{y}{3} + \frac{y}{3},$$

so t_2 is the order $m_2 = 3m-1$ and since $m \in [\frac{1}{3}, \frac{2}{3}]$, then $m_2 \in [0, 1]$. Solving $m_2 = \frac{1}{3}$ and $m_2 = \frac{2}{3}$ we conclude the desired statement.

Finally, we look at T_3 and obtain

$$x + \frac{y}{2} < x + \frac{y}{2} + (\frac{1}{3})\frac{y}{2} < x + \frac{y}{2} + \frac{y}{2},$$

consequently T_3 is type II with order $m_3 = \frac{1}{3}$. □

4.3. **Boundary conditions of type III.**

Lemma 4.5. *If $T = \{p_1, p_2, p_3\}$ is type III of order $m \in (\frac{2}{3}, 1]$ we obtain*

(1) $T_1 = \{p_1, q_3, q_2\}$ is type III of order
$$m_1 = m.$$
(2) $T_2 = \{q_3, q_1, p_2\}$ verifies
(a) If $m \in (\frac{2}{3}, \frac{6}{7}]$ then T_2 is type II.
(b) If $m \in (\frac{6}{7}, 1]$ then T_2 is type III.
In both cases the order of T_2 is
$$m_2 = \frac{2-m}{2m}.$$
(3) $T_3 = \{q_2, q_1, p_3\}$ is type II of order
$$m_3 = \frac{m}{2}.$$

In this case it is important to note that there is also a rotation of the triangle T_2.

Proof. In this case we obtain
$$f(q_1) = x + (\frac{m+2}{4})y, \quad f(q_2) = x + \frac{1}{2}y, \text{ and } f(q_3) = x + \frac{m}{2}y.$$

Now, for T_1 we have
$$x < x + m\frac{y}{2} < x + \frac{y}{2},$$
so T_1 is type III of order $M_1 = m$.

On T_2 we have
$$x \le x + \frac{m}{2}y \le x + \frac{m}{2}y + (\frac{2-m}{2m})\frac{m}{2}y \le x + \frac{m}{2}y + \frac{m}{2}y,$$
therefore T_2 is of order $m_2 = \frac{m-2}{2m}$ the proof of this case follows solving the equation $m_2 = \frac{2}{3}$ to obtain the borderline value $m = \frac{6}{7}$.

Finally, for T_3 we get
$$x + \frac{y}{2} \le x + \frac{y}{2} + \frac{m}{2}\frac{y}{2} \le x + \frac{y}{2} + \frac{y}{2},$$
then T_3 is type II of order $m_3 = \frac{m}{2}$. □

Remark 4.6. We have assumed that the minimum and maximum value of the initial data of our problem occur at p_1 (upper vertex) an p_3 (lower right vertex), respectively. When this holds, we will say that the triangle is well ordered. This fails to hold in T_2 for both types I and II but holds in the rest of the cases. That T_1 and T_3 always verify this property will help us to prove that the solution is linear on the side of the triangle that joins p_1

with p_3, $L_{p_1p_3}$. To prove this result will be our main goal in the remaining of this section. We start this analysis with an elementary lemma.

Lemma 4.7. *Let $f:[0,1]\to\mathbb{R}$ be a continuous function such that $f(0)=a$, $f(1)=b$ and*

$$f(\frac{k}{2^p})=\frac{f(\frac{k-1}{2^p})+f(\frac{k+1}{2^p})}{2}\qquad \forall k\in\{1,3,\dots,2^p-1\},\forall p\in\mathbb{N}.$$

Then $f(x)=(b-a)x+a$.

Proof. Let p be a nonnegative integer. We denote

$$D_p=\left\{0,\frac{1}{2^p},\dots,\frac{2^p-1}{2^p},1\right\} \tag{4.8}$$

and we write $g(x)=(b-a)x+a$. If we prove that

$$f(x)=g(x),\quad \forall x\in\cup_{p\geq 0}D_p \tag{4.9}$$

then by continuity, using that $\cup_{p\geq 0}D_p$ is dense in $[0,1]$, we conclude that

$$f(x)=g(x)\quad \forall x\in[0,1]. \tag{4.10}$$

Hence, our goal is to show by induction (4.9).

The case $p=0$ is trivially true. If $p=1$ we just have to prove that $f(\frac{1}{2})=g(\frac{1}{2})$ but we have

$$f(\frac{1}{2})=\frac{f(0)+f(1)}{2}=\frac{a+b}{2}=(b-a)\frac{1}{2}+a=g(\frac{1}{2}).$$

Now we suppose that (4.9) holds for $p-1$. Let $x=\frac{k}{2^p}\in D_p$. If k is even, then $x\in D_{p-1}$ and we use the inductive hypothesis, so we suppose that k is odd and so both $k-1$ and $k+1$ are even. Then,

$$\begin{aligned}f(\frac{k}{2^p})&=\frac{f(\frac{k-1}{2^p})+f(\frac{k+1}{2^p})}{2}=\frac{g(\frac{k-1}{2^p})+g(\frac{k+1}{2^p})}{2}\\&=\frac{1}{2}[(b-a)\frac{k-1}{2^p}+a+(b-a)\frac{k+1}{2^p}+a]\\&=(b-a)\frac{k}{2^p}+a=g(\frac{k}{2^p}).\end{aligned}$$

This ends the proof. □

Proposition 4.11. *Let f be the solution in SG satisfying the boundary conditions $f(p_1)\leq f(p_2)\leq f(p_3)$, now we denote by $L_{p_1p_3}$ the segment with endpoints p_1 and p_3, that is*

$$L_{p_1p_3}=\{p_1+t(p_3-p_1):t\in[0,1]\},$$

then if we define $h:[0,1]\to\mathbb{R}$ by $h(t)=f(p_1+t(p_3-p_1))$ we have that

$$h(t)=f(p_1)+(f(p_3)-f(p_1))t.$$

That is, f is lineal on $L_{p_1p_3}$.

Proof. First, we denote $L_{p_1p_3}(t) = p_1 + t(p_3 - p_1)$, $t \in [0,1]$, we observe that $h(0) = f(L_{p_1p_3}(0)) = f(p_1)$ and $h(1) = f(L_{p_1p_3}(1)) = f(p_3)$. Now we have that $q_2 = L_{p_1p_3}(\frac{1}{2})$ and thanks to (3.1) we obtain

$$f(q_2) = \frac{f(p_1) + f(p_3)}{2} = h(q).$$

Thus f and h are equal on D_1.

Now using Lemmas 4.3, 4.4 and 4.5, the triangles T_1 and T_3 are well ordered and therefore in each new iteration the value of the new point generated on the segment $L_{p_1p_3}$ is the arithmetic mean of the values of the vertices that were already in it.

Finally, we have that these new points are the image by $L_{p_1p_3}$ of some element of $\cup_{p\geq 0} D_p$. Now, we just use Lemma 4.7 to conclude. □

Lemma 4.12. *Let f and g be two solutions in SG satisfying the boundary conditions $f(p_1) \leq f(p_2) \leq f(p_3)$ and $g(p_1) \leq g(p_2) \leq g(p_3)$ respectively. If we suppose that these boundary conditions are of the same order, then, there exists a function h, defined in SG such that*

$$h(f(x)) = g(x), \quad \forall x \in SG.$$

Proof. We put $f(p_1) = a_1$, $f(p_2) = a_1 + b_1$, $f(p_3) = a_1 + mb_1$ and $g(p_1) = a_2$, $g(p_2) = a_2 + b_2$, $g(p_3) = a_2 + mb_2$, with $b_1, b_2 > 0$. Let

$$h(x) = \frac{1}{b_1}(b_2 x + b_1 a_2 - b_2 a_1).$$

This function verifies,

$$h(f(p_i)) = g(p_i), i = 1, 2, 3.$$

We will see that $h \circ f$ coincide with g on $V_1 \setminus V_0 = \{q_1, q_2, q_3\}$, regardless of the type of boundary conditions, therefore they coincide on V_* and for the continuity on SG.

First, we observe that

$$f(q_2) = a_1 + \frac{m}{2} b_1$$

and

$$h(f(q_2)) = a_2 + \frac{m}{2} b_2 = g(q_2).$$

On the other hand we have

$$f(q_3) = \frac{1}{2} a_1 + \frac{1}{2} \max\{a_1 + mb_1, f(q_1)\},$$

$$f(q_1) = \frac{1}{2}(a_1 + b_1) + \frac{1}{2} \min\{a_1 + mb_1, f(q_3)\},$$

Now as h is an increasing function we have

$$h(\max\{a_1 + mb_1, f(q_1)\}) = \max\{h(a_1 + mb_1), h(f(q_1))\} = h((2f(q_3) - a_2),$$

and since

$$h((2f(q_3)-a_2)=\frac{1}{b_1}(2b_2f(q_3)-b_2a_1+b_1a_2-b_2a_1)=2h(q_3)-a_2,$$

we obtain

$$h(q_3)=\frac{1}{2}a_1+\frac{1}{2}\max\{a_2+mb_2,h(f(q_1))\}$$

In the same way we have that

$$h(q_1)=\frac{1}{2}(a_2+b_2)+\frac{1}{2}\min\{a_2+mb_2,h(f(q_3))\},$$

therefore, $g(q_1)=h(f(q_1))$ and $g(q_3)=h(f(q_3))$. □

This lemma implies that if we have a triangle T with boundary conditions of type $m\in[0,1]$, we can obtain the values on T if we know the values for a triangle T with boundary condition of the form $0,m,1$. We will say that all the triangles of the same type are equivalent in this sense.

Now we could look if there is any sort of order in the type of the subsequent triangles. This is, if we start with a triangle of type m, is it possible obtain any information about the next triangles ?. We begin with the following result.

Corollary 4.13. *Let f be the solution on SG with initial conditions $f(p_1)\le f(p_2)\le f(p_3)$. If there exists a triangle T_i, $i=1,2,3$ such that T and T_i are of the same type, then it is possible obtain the values of f on T_i, knowing the values of f on T_j for all $j\neq i$.*

Proof. We may assume, without loss of generality, that $i=1$. Then T_1 is the triangle of vertices $\{F_1(p_1),F_1(p_2),F_1(p_3)\}$, but the order of this T_1 is the same as the order of T so if we know the values of f $(T_2\cup T_3)\cap SG$, thanks to Lemma 4.12 we know the values of f on $F_2(F_1(T))\cup F_3(f_1(T)\cap SG$. Repeating this process we obtain the values of f on all points on $F_1(T)\cap SG$. □

Now, we need a series of lemmas that tell us how to use the previous result.

Lemma 4.14. *Let f be the solution on SG with boundary conditions $f(p_1)=f(p_2)=a<f(p_3)=a+b$, b a positive number. The function f is determined by $f(T_1)$.*

Proof. We know, by Lemma 4.3, that T_1 and T_2 are both type II of order $\frac{1}{2}$ and furthermore, T_3 and T are type III of the same order. Moreover, T_1 and T_2 have symmetrical values on its vertex so, if we know f on T_1 we obtain the values of f on T_2. Now, if we apply Lemma 4.13 we conclude the result. □

Lemma 4.15. *Let f be the solution on SG with boundary conditions $f(p_1) = a < f(p_2) = a + \frac{1}{2}b < f(p_3) = a + b$, b a positive number. Then, the function f is determined by $f(T_1)$ and $f(T_3)$.*

Proof. Thanks to Lemma 4.4, T_1 and T_3 are both type II of order $\frac{2}{3}$ and $\frac{1}{3}$ respectively and T_2 and T are both type II with order $\frac{1}{2}$. □

Lemma 4.16. *Let f be the solution on SG with boundary conditions $f(p_1) = a < f(p_2) = a + \frac{2}{3}b < f(p_3) = a + b$, b a positive number. Then, the function f is determined by $f(T_1)$ and $f(T_2)$.*

Proof. Thanks to Lemma 4.4, T_2 is type II of order 1, T_3 is type II of order $\frac{1}{3}$, and T_1 and T are both type II of order $\frac{2}{3}$. □

Lemma 4.17. *Let f be the solution on SG with boundary conditions $f(p_1) = a < f(p_2) = a + \frac{1}{3}b < f(p_3) = a + b$, b a positive number. The function f is determined by $f(T_1)$ and $f(T_2)$.*

Proof. Thanks to Lemma 4.4, T_1 is type II of order $\frac{2}{3}$, T_2 is type I of order 0, and T_3 and T are both type II of order $\frac{1}{3}$. □

Thanks to Lemmas 4.17 and 4.16, we need to know the behavior of the triangles of type 1.

Lemma 4.18. *Let f be the harmonic function on SG with initial conditions $f(p_1) = a < f(p_2) = a+b < f(p_3) = a+b$, b a positive number. The function f is determined by $f(T_2)$ and $f(T_3)$.*

Proof. Thanks to lemma 4.5 T_1 and T_2 are both type II of order $\frac{1}{2}$ and T_3 and T are both type III of order 1. □

Corollary 4.19. *Let m be a number in the set $M = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$, If T is a triangle with initial conditions m then all subtriangles are type n with $n \in M$.*

5. Continuous dependence

Let $f(p_1)$, $f(p_2)$ and $f(p_3)$ be three positive real numbers, and suppose that $(g_n(p_1))_n$, $(g_n(p_2))_n$ and $(g_n(p_3))_n$ are three sequences of positive real numbers converging to $f(p_1)$, $f(p_2)$ and $f(p_3)$ respectively.

For each $n \in \mathbb{N}$ let g_n be the solution on SG satisfying the boundary conditions $g_n(p_1)$, $g_n(p_2)$ and $g_n(p_3)$. We know that g_n is Lipschitz continuous on SG with constant less or equal than $2C_n = 2\max\{|g_n(p_i) - g_n(p_j)| : i, j \in \{1, 2, 3\}\}$.

Lemma 5.1. *In above conditions we have*

(1) The sequence $(g_n)_n$ is uniformly bounded on SG.

(2) The sequence $(g_n)_n$ is uniformly Lipschitz continuous on SG.

Proof. Since the sequences $(g_n(p_i))_n$ converge, for $i = 1, 2, 3$ there exists $M_i = \max\{g_n(q_i) : n \in \mathbb{N}\}$ and $m_i = \min\{g_n(q_i) : n \in \mathbb{N}\}$. We denote by

$$M = \max\{f(p_1), f(p_2), f(p_3), M_1, M_2, M_3\},$$

and

$$m = \min\{f(p_1), f(p_2), f(p_3), m_1, m_2, m_3\}.$$

Proof of (1). For all $n \in \mathbb{N}$ and for all $x \in SG$ we have $g_n(x) \leq \max\{g_n(q_1)_n, g_n(q_2), g_n(q_3)\}$ so $g_n(x) \leq M$ for all $x \in SG$. Therefore, $(g_n)_n$ is uniformly bounded.

Proof of (2). We know that for all $n \in \mathbb{N}$, g_n is Lipschitz continuous with constant less or equal than

$$2C_n = 2\max\{|g_n(q_i) - g_n(q_j)| : 1 \leq i, j \leq 3\} \leq 2(M - m).$$

Then $(g_n)_n$ is uniformly Lipschitz continuous on SG. □

Proof of Theorem C. Thanks to Lemma 5.1 we obtain that the sequence $(g_n)_n$ is uniformly bounded and equicontinuous (since it is uniformly Lipschitz continuous) on SG, then we can apply Ascoli-Arzela Theorem's and get a subsequence (that we still denote by $(g_n)_n$) converging uniformly to a Lipschitz continuous function g on SG.

Now, we take three different indexes i, j and k in $\{1, 2, 3\}$ and we denote by A_i the set $\{f(p_j), f(p_k), g_n(q_j), g_n(q_k)\}$. We have

$$g(q_i) = \lim_{n\to+\infty} g_n(q_i) = \lim_{n\to+\infty} \frac{1}{2}\left(\max A_i + \min A_i\right).$$

Now if we observe that $\lim_{n\to+\infty} A_i = \{f(p_j), f(p_k), g(q_j), g(q_k)\} := A^i$ we obtain

$$g(q_i) = \frac{1}{2}\left(\max A^i + \min A^i\right).$$

Hence, from our contraction we obtain

$$g(q_i) = f(q_i).$$

That is, we obtain (arguing by induction) that g is a solution to our problem with boundary values $f(p_1)$, $f(p_2)$ and $f(p_3)$. The proof of the result follows thanks to our existence and uniqueness result. In fact, f and g are two solutions with boundary values $f(p_1)$, $f(p_2)$ and $f(p_3)$ and then we conclude that

$$f = g = \lim_{n\to\infty} g_n.$$

Since we proved that every sequence has a subsequence that converges to a unique limit, we obtain the convergence of the whole sequence $(g_n)_n$ to f, as we wanted to show. □

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